Integrerend project systeemtheorie

21/01/2013, Monday, 9:00-12:00

1 $(4 + 4 + 8 + 4 = 20)$ **Linearization**

Consider the so-called Van der Pol system

$$\ddot{z}(t) - \mu\big(1 - z^2(t)\big)\dot{z}(t) + z(t) = 0.$$

(a) Write the system in the form of a nonlinear state-space system ($\dot{x} = f(x)$) by taking $x_1(t) = z(t)$ and $x_2(t) = \dot{z}(t)$.

(b) Show that $x_1(t) = x_2(t) = 0$ is a solution of $\dot{x} = f(x)$.

(c) Determine the linearized system.

(d) For which values of μ is the linearized system asymptotically stable.

2 (15) **Routh criterion**

Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -a \end{bmatrix} x$$

where a is real number. For which values of a is this system asymptotically stable?

3 $(3 + 4 + 4 + 4 + 4 + 4 + 8 + 8 = 35)$ **Controllability and observability**

Consider the linear system

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

Explain your answers to the following questions:

(a) Is it stable?

(b) Is it controllable?

(c) Is it observable?

(d) Is it stabilizable?

(e) Is it detectable?

(f) Does there exist an observer of the form $\dot{\hat{x}} = P\hat{x} + Qu + Ry$?

(g) Does there exist a stabilizing dynamic compensator (from y to u)? If yes, determine such a compensator.

4 $(3 + 5 + 12 = 20)$ **Detectability**

Consider the linear systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^m$ is the output. Let $x(t, x_0)$ denote the state trajectory of the system corresponding to the initial condition x_0. Define

$$W = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

and

$$\mathcal{V} = \{x_0 \mid \lim_{t\to\infty} x(t, x_0) = 0\}.$$

Show that

(a) $\mathcal{V}$ is a subspace.

(b) $\mathcal{V}$ is A-invariant.

(c) if $\ker W \subseteq \mathcal{V}$ then the system is detectable.

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10 pts gratis.

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(1) Van der Pol system is given by

$$\ddot{z} - \mu[1 - z^2]\dot{z} + z = 0.$$

(a) Take

$$x_1 = z$$
$$x_2 = \dot{z}.$$

Then, we have

$$\dot{x}_1 = \dot{z} = x_2$$
$$\dot{x}_2 = \ddot{z} = -z + \mu[1 - z^2]\dot{z} = -x_1 + \mu[1 - x_1^2]x_2.$$

Hence, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \mu(1 - x_1^2)x_2 \end{bmatrix}. \qquad (*)$$

(b) If $x_1(t) = x_2(t) = 0$, then $(*)$ is obviously satisfied.

(c) The linearized system is given by $\dot{\hat{x}} = \left.\frac{\partial f}{\partial x}\right|_{x=0} \hat{x}$ where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -x_1 + \mu(1 - x_1^2)x_2 \end{bmatrix}.$$

Note that

$$\frac{\partial f}{\partial x}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1-2\mu x_1 x_2 & \mu(1-x_1^2) \end{bmatrix}$$

Then, we obtain

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \hat{x} \qquad (**)$$

for the linearized system.

(d) Characteric polynomial of $\begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$ is given by

$$\det\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda-\mu \end{bmatrix}\right) = \lambda^2 - \mu\lambda + 1.$$

Corresponding Routh table can be found as:

$$\begin{matrix} 1 & 1 \\ -\mu & \\ 1 & \end{matrix}$$

Then, $(**)$ is asymptotically stable if and only if $\mu < 0$.

Alternatively, one can explicitly write down the roots of the characteristic polynomial as

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

and conclude that $(**)$ is asy. stable if and only if $\mu < 0$.

(2) Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -a \end{bmatrix} x = Ax$$

where a is real number. Note that

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 3 & 4 & \lambda + a \end{bmatrix}\right)$$

$$= \lambda^3 + a\lambda^2 + 4\lambda + 3.$$

The corresponding Routh table can be found as:

$$\begin{array}{cc} 1 & 4 \\ a & 3 \\ \frac{4a-3}{a} & \\ 1 & \end{array}$$

Therefore, the system is asy. stable if and only if $a > \frac{3}{4}$.

(3) In this problem, a linear system of the form

$$\dot{x} = Ax + Bu \qquad y = Cx$$

is given where

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = [0 \;\; 1].$$

(a) The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -3$. Since A has an eigenvalue, namely $\lambda_1 = 2$, having a positive real part, the system is NOT stable.

(b) The controllability matrix is given by

$$R = [B \;\; AB] = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}.$$

Since rank $R = 1$, the system is NOT controllable.

(c) The observability matrix is given by

$$W = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}.$$

Since rank $W = 2$, the system is observable.

(d) The system is stabilizable if and only if

$$\text{rank}\,[\lambda I - A \;\; B] = n$$

for all eigenvalues λ of A with $Re(\lambda) \geq 0$. Therefore, we need to check this condition only for $\lambda_1 = 2$. Note that

$$[\lambda_1 I - A \;\; B] = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 5 & 1 \end{bmatrix}.$$

Since $\text{rank}\,[\lambda_1 I - A \;\; B] = 1$, the system is NOT stabilizable.

(e) The system is detectable since it is already observable as it was shown in (c). Alternatively, one can use the characterization of detectability:

The system is detectable if and only if

$$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$$

for all eigenvalues of A with $Re(\lambda) \geq 0$. Since

$$\text{rank} \begin{bmatrix} \lambda_1 I - A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 0 \\ -1 & 5 \\ 0 & 1 \end{bmatrix} = 2,$$

the system is detectable.

(f) There exists an observer in the given form if and only if (C, A) is detectable. Therefore, it follows from (e) that there exists such an observer.

(g) There exists a dynamic compensator which stabilizes the system if and only if (A, B) is stabilizable and (C, A) is detectable. Therefore, it follows from (d) that there does NOT exist such a compensator.

④ Note that

$$x(t, x_0) = e^{At} x_0.$$

Then, we have

$$\mathcal{V} = \{x_0 \mid \lim_{t\to\infty} e^{At} x_0 = 0\}.$$

ⓐ We know that $\mathcal{V}$ is a subspace if and only if

$$a_1 x_1 + a_2 x_2 \in \mathcal{V} \text{ for all } x_1, x_2 \in \mathcal{V} \text{ and } a_1, a_2 \in \mathbb{R}.$$

Since

$$\lim_{t\to\infty} e^{At}(a_1 x_1 + a_2 x_2) = \lim_{t\to\infty} a_1 e^{At} x_1 + a_2 e^{At} x_2 = 0$$

for all $x_1, x_2 \in \mathcal{V}$ and $a_1, a_2 \in \mathbb{R}$, we can conclude that $\mathcal{V}$ is a subspace.

ⓑ Let $x_0 \in \mathcal{V}$. Then, we have

$$\lim_{t\to\infty} e^{At} x_0 = 0. \qquad (*)$$

Note that $e^{At} A x_0 = A e^{At} x_0$ since A and e^{At} commute. Therefore, we have

$$\lim_{t\to\infty} e^{At}(A x_0) = \lim_{t\to\infty} A e^{At} x_0 \overset{(*)}{=} 0.$$

This means that $A x_0 \in \mathcal{V}$ and hence $\mathcal{V}$ is A-invariant.

ⓒ The system is detectable if and only if

$$\operatorname{rank}\begin{bmatrix}\lambda I - A\\ C\end{bmatrix} = n \quad (*)$$

for all $\lambda \in \mathbb{C}$ with $Re(\lambda) \geq 0$. Therefore, it is enoug to show that $\ker W \subseteq \mathcal{V}$ implies $(*)$. Let z be such that

$$\begin{bmatrix}\lambda I - A\\ C\end{bmatrix} z = 0$$

for some $\lambda \in \mathbb{C}$ with $Re(\lambda) \geq 0$. Then, we have

$$Az = \lambda z \text{ and } Cz = 0.$$

Note that $CA^k z = \lambda^k Cz = 0$ for all $k = 0, 1, \ldots$. Hence, we get that $z \in \ker W$. Since $\ker W \subseteq \mathcal{V}$, z must belong to $\mathcal{V}$. In other words,

$$\lim_{t\to\infty} e^{At} z = 0. \quad (**)$$

Note that $e^{At} z = \left(I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots\right) z = \left(1 + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + \cdots\right) z = e^{\lambda t} z$ since $Az = \lambda z$. Then, it follows from $(**)$ that

$$\lim_{t\to\infty} e^{\lambda t} z = 0.$$

Since $Re(\lambda) \geq 0$, this is possible only if $z = 0$. Therefore, $(*)$ holds.